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ON THE CLASSIFICATION OF THREE-DIMENSIONAL COMPACT KAEHLER MANIFOLDS OF NONNEGATIVE BISECTIONAL CURVATURE

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0. Introduction

After the solution of Frankel conjecture by Mori [5] and Siu & Yau [8], it is natural to consider the classification of compact Kaehler manifolds of nonnegative bisectional curvature. In this direction there are some previous works, for example, the characterization of hyperquadrics by Siu [7], and the splitting theorem of Kaehler manifolds of nonnegative bisectional curvature by Howard, Smyth, and Wu [3], [9]. Besides these general dimensional studies there is a low dimensional result by Howard & Smyth [2] that is the complete classification of two-dimensional compact Kaehler manifolds of nonnegative curvature. In this paper, proceeding in this direction, we consider the case of three-dimension and obtain some results which, combined together with the above results of Howard, Smyth and Wu, [2], [3] and [9], enable us to settle the classification of three-dimensional compact Kaehler manifolds of nonnegative bisectional curvature. Namely our goal is the following theorem.

Theorem 3. *Let M be a three-dimensional compact Kaehler manifold of nonnegative bisectional curvature. If M has quasipositive Ricci curvature, then M is biholomorphic to one of the following*: P^3, Q^3, $P^1 \times P^2$, $P^1 \times P^1 \times P^1$.

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1. Notations

Let M be an n-dimensional Kaehler manifold with a Kaehler metric g. We can define the holomorphic tangent bundle TM and the antiholomorphic

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tangent bundle $\overline{TM}$. The complexified tangent bundle is the direct sum of TM and $\overline{TM}$. We can extend the metric g, the Riemannian curvature tensor R and the Ricci curvature tensor Ric to be forms on the complexified tangent bundle. Then using a local coordinate system $z^1, z^2, \cdots, z^n$, we have that

$$
\begin{aligned}
& g_{ij} = g_{\overline{ij}} = 0, \qquad \overline{g_{i\bar{j}}} = g_{\bar{i}j} = g_{j\bar{i}}, \\
& \partial g_{i\bar{j}}/\partial z^k = \partial g_{k\bar{j}}/\partial z^i, \\
& R_{i\bar{j}k\bar{l}} = -\partial^2 g_{i\bar{j}}/\partial z^k \partial \overline{z^l} + g^{p\bar{q}} \partial g_{i\bar{q}}/\partial z^k \partial g_{p\bar{j}}/\partial \overline{z^l}, \\
& R_{i\bar{j}} = \mathrm{Ric}_{i\bar{j}} = R_{i\bar{j}p\bar{p}} = -\partial^2 \log \det(g_{k\bar{l}})/\partial z^i \partial \overline{z^j},
\end{aligned}
$$

where we used the summation convention. Note that our sign convention of the curvature is different from the usual one.

For unit vectors $X, Y \in T_pM$, $R(X, \overline{X}, Y, \overline{Y})$ is called the bisectional curvature in the direction (X, Y).

M is said to be of nonnegative (positive) bisectional curvature if for all pairs of unit vectors $X, Y \in T_pM$, the bisectional curvature in the direction is nonnegative (positive, respectively) everywhere.

We say that Ricci curvature is quasipositive if Ricci tensor is positive semidefinite everywhere and positive definite somewhere.

We define $N(X)$, $\tilde{N}(X)$ for nonzero $X \in T_pM$ as follows:

$$
\begin{aligned}
N(X) &= \left\{ Y \in T_pM \middle| R(X, \overline{X}, Y, \overline{Y}) = 0 \right\}, \\
\tilde{N}(X) &= \left\{ Y \in T_pM \middle| R(X, \overline{X}, Y, \overline{Y}) = 0,\ Y \perp X \right\}.
\end{aligned}
$$

Note that $N(X)$, $\tilde{N}(X)$ are complex linear subspaces of T_pM if M is of nonnegative bisectional curvature.

For M of nonnegative bisectional curvature we define the condition (C) at p as follows:

Condition (C) *at p*. If $T_pM = H_1 \oplus H_2$ is an orthogonal decomposition and $0 \neq X_i \in H_i$ $(i = 1, 2)$, then either $H_1 \not\subset N(X_2)$ or $H_2 \not\subset N(X_1)$.

Remark that in the definition of the condition (C) we can use $\tilde{N}(X)$ instead of $N(X)$.

2. Summary of previous results

In their paper [2], Alan Howard and Brian Smyth gave the complete classification of compact Kaehler surfaces of nonnegative bisectional curvature.

Theorem A. *Let M be a compact Kaehler surface of nonnegative bisectional curvature. Then one of the following holds.*

(1) *M is biholomorphic to the projective space P^2.*

(2) *M is biholomorphic to $P^1 \times P^1$ and the metric is a product of metrics of nonnegative curvature.*

(3) *M is flat.*

(4) *M is a ruled surface (i.e. P^1-bundle) over an elliptic curve. In this case the universal covering space of M is $\mathbf{C}^1 \times P^1$ endowed with the product of the flat metric on $\mathbf{C}^1$ and a metric of nonnegative curvature on P^1.*

And if the Ricci tensor is positive at some point, then (1) *or* (2) *holds.*

Siu proved in his paper [7] that we can characterize the complex projective space and the complex hyperquadric by the properties of curvature.

Theorem B. *Let M be an n-dimensional ($n \geqslant 3$) compact complex Kaehler manifold of nonnegative bisectional curvature. Suppose that M satisfies the condition* (C) *everywhere and has the following property* $(*)$ *at some point $p \in M$:*

$$(*) \qquad \dim_{\mathbf{C}} N(X) \leqslant 1 \quad \text{for all nonzero } X \in T_pM.$$

Then M is biholomorphic to either the projective space P^n or the hyperquadric Q^n.

There are other words of more general nature on the classification problem by Alan Howard, Brian Smyth, and H. Wu [3], [9].

Theorem C. *Let M be an n-dimensional compact Kaehler manifold of nonnegative bisectional curvature and let the maximum rank of* Ric *on M be $n - k$ $(0 \leqslant k \leqslant n)$. Then*:

(A) *The universal covering of M is holomorphically isometric to a direct product $M' \times \mathbf{C}^k$, where M' is an $(n-k)$-dimensional compact Kaehler manifold with quasipositive Ricci curvature and $\mathbf{C}^k$ is equipped with the flat metric.*

(B) *M' is algebraic, possesses no nonzero holomorphic q-forms for $q \geqslant 1$, and is holomorphically isometric to a direct product of compact Kaehler manifolds $M_1 \times \cdots \times M_s$, where each M_i has quasipositive Ricci curvature and satisfies $H^2(M_i, \mathbf{Z}) \cong \mathbf{Z}$.*

(C) *There is a flat, compact manifold B and a holomorphic, locally isometric trivial fibration p: $M \to B$ whose fibre is M'.*

(D) *There exist a compact Kaehler manifold M^*, a flat complex torus T and a commutative diagram*

$$\begin{array}{ccc} M^* & \longrightarrow & T \\ \downarrow & & \downarrow \\ M & \longrightarrow & B \end{array}$$

where the horizontal maps are holomorphic, locally isometrically trivial fibrations with fibre M', and the vertical maps are finite coverings. Furthermore, M^* is globally diffeomorphic to $M' \times T$.

In particular, $\pi_1(M)$ is either trivial or an infinite crystallographic group.

By the above theorems, to settle the classification of three-dimensional compact Kaehler manifolds of nonnegative bisectional curvature, it is sufficient to prove it in the case of quasipositive Ricci curvature. We will prove that such manifolds are biholomorphic to one of the following: P^3, Q^3, $P^1 \times P^2$, $P^1 \times P^1 \times P^1$.

3. Generalization of Siu's theorem

In this section we slightly generalize Siu's theorem stated in the previous section.

Theorem 1. *Let M be an n-dimensional ($n \geqslant 3$) compact Kaehler manifold of nonnegative bisectional curvature which satisfies the condition* (C) *at every point. If there exists some point $p \in M$ such that*

$$(**) \qquad \dim_{\mathbf{C}} \tilde{N}(X) \leqslant 1 \quad \textit{for all nonzero } X \in T_pM,$$

then M is biholomorphic to either P^n or Q^n.

Corollary. *Let M be an n-dimensional ($n \geqslant 3$) compact Kaehler manifold of nonnegative bisectional curvature with the property* $(**)$ *satisfied everywhere. Then, M is biholomorphic to either P^n or Q^n.*

Proof. It is easy to see that the property $(**)$ at p implies the condition (C) at p.

Since the proof of the theorem is the same as that of Siu's, except at the points where the property $(*)$ is used, we only point out the places where changes must be done and write the altered argument. Siu used the property $(*)$ twice. First, he used it to prove the following fact.

Fact. Let $f\colon P^1 \to M$ be a nonconstant holomorphic map such that $f(0) = p$, where $p \in M$ is the point that satisfies $(*)$. One can decompose the induced holomorphic vector bundle f^*TM into n holomorphic line bundles,

$$f^*TM = L_1 \oplus L_2 \oplus \cdots \oplus L_n.$$

Then we conclude that the first Chern class $C_1(L_i) > 0$ except at most one L_i.

We prove this fact using the property $(**)$ instead of the property $(*)$. Let us denote the induced metric and the connection on the dual bundle $(f^*TM)^*$ by $\langle\ ,\ \rangle$ and ∇'. Then the connection ∇'' of the dual bundle L_i^* of L_i can be written as

$$\nabla'_X u = \nabla''_X u + a(X, u),$$

where u is a section of L_i^* and a is the second fundamental form, which is orthogonal to L_i^*. We compute the curvature R'' of L_i^* with respect to the induced metric in terms of the curvature R' of $(f^*TM)^*$. For $X \in TP^1$,

$$\begin{aligned} R'(X, \bar{X})u &= -\nabla'_X \nabla'_{\bar{X}} u + \nabla'_{\bar{X}} \nabla'_X u + \nabla'_{[X, \bar{X}]} u \\ &= -\nabla'_X(\nabla''_{\bar{X}} u) + \nabla'_{\bar{X}}(\nabla''_X u + a(X, u)) \\ &\quad + \nabla''_{[X, \bar{X}]} u + a([X, \bar{X}], u) \\ &= -\nabla''_X(\nabla''_{\bar{X}} u) - a(X, \nabla''_{\bar{X}} u) + \nabla''_{\bar{X}} \nabla''_X u \\ &\quad + \nabla'_{\bar{X}} a(X, u) + \nabla''_{[X, \bar{X}]} u + a([X, \bar{X}], u), \\ R'(u, \bar{u}, X, \bar{X}) &= -\langle R'(X, \bar{X})u, \bar{u} \rangle \\ &= -\langle R''(X, \bar{X})u, \bar{u} \rangle - \langle \nabla'_{\bar{X}}(a(X, u)), \bar{u} \rangle \\ &= -\langle R''(X, \bar{X})u, \bar{u} \rangle + \langle a(X, u), \overline{\nabla'_X u} \rangle \\ &= R''(u, \bar{u}, X, \bar{X}) + \langle a(X, u), \overline{a(X, u)} \rangle. \end{aligned}$$

Let the section Y of f^*TM be the metric dual of u such that

$$v(Y) = \langle v, \bar{u} \rangle \quad \text{for all sections } v \text{ of } (f^*TM)^*.$$

Then we have that

$$R'(u, \bar{u}, X, \bar{X}) = -R(Y, \bar{Y}, f_* X, \overline{f_* X}),$$

where R is the curvature tensor of M. Thus we get that the curvature of L_i^* is nonpositive, especially $C_1(L_i) \geqslant 0$. Moreover we know that if $C_1(L_i) = 0$, L_i^* is a trivial line bundle and the bisectional curvature in the direction of $(f_* TP^1, Y)$ is zero. If L_i^* is a trivial bundle there exists a nonvanishing holomorphic section u of L_i^*. Let X be any holomorphic tangent vector field on P^1. Then $u(f_* X)$ makes a holomorphic function on P^1, and X vanishes somewhere. We get that $u(f_* X) = 0$, which means that $f_* X$ and the metric dual Y of u are perpendicular to each other. Therefore if we have two L_i with the vanishing first Chern class we get a contradiction near p. (Note that the points which satisfy (**) make an open set.)

Another case in which Siu used the property (*) is very similar: Let f be a nonconstant holomorphic map as above, and E be the divisor of df. Then $TP^1 \otimes [E] \subset f^*TM$, and we get a decomposition

$$f^*TM/TP^1 \times [E] = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_{n-1}.$$

Siu claimed that $C_1(Q_i) > 0$ with at most one exception. We can prove this similarly. Thus we get the theorem.

4. Hamilton's equation

In his paper [1], Richard S. Hamilton introduced an evolution equation in metric tensors and proved the short time solvability. He considered it in the real three-dimensional case and proved that the equation preserves nonnegativity of Ricci curvature. In this section we consider the equation in the case of complex dimension three, and show that it preserves nonnegativity of bisectional curvature.

First we remark that, as he stated, the equation preserves Kaehlerity of the initial metric. An easy way to see this is to reduce the equation to a parabolic equation on functions in the following way. Let g be a given Kaehler metric and $R_{i\bar{j}}$ be its Ricci tensor. We solve the following parabolic equation on a function u:

$$du/dt = \log\{\det(g_{i\bar{j}} - tR_{i\bar{j}} + u_{i\bar{j}})/\det(g_{i\bar{j}})\}, \qquad u|_{t=0} = 0,$$

where $u_{i\bar{j}}$ means $\partial^2 u/\partial z^i \partial \overline{z^j}$. Then the Kaehler metric $(g_{i\bar{j}} - tR_{i\bar{j}} + u_{i\bar{j}})$ gives a solution metric of Hamilton's equation,

$$dg_{i\bar{j}}/dt = -R_{i\bar{j}}.$$

Moreover because of the parabolicity it is easy to see that the equation has the solvability for a short time.

Theorem 2. *Let M be a three-dimensional compact Kaehler manifold. Then Hamilton's equation preserves nonnegativity of bisectional curvature. Moreover if the bisectional curvature of the initial metric is positive at some point, the solution metric has positive bisectional curvature everywhere for $t > 0$.*

In the proof we use a proposition which is almost the same as Hamilton's, but since there was a slight gap in his proof, we give a proof of the proposition for the sake of completeness. First we define some notations on a Kaehler manifold M.

Definition. Let u, v be tensors which have the same type and the same symmetric properties as curvature tensor. We say that $u_p \geqslant v_p$ ($u_p > v_p$) for $p \in M$, if, for all nonzero $X, Y \in T_pM$,

$$u_p(X, \overline{X}, Y, \overline{Y}) \geqslant v_p(X, \overline{X}, Y, \overline{Y}),$$
$$\big(u_p(X, \overline{X}, Y, \overline{Y}) > v_p(X, \overline{X}, Y, \overline{Y}), \text{ respectively}\big),$$

and that $u \geqslant v$ ($u > v$) if $u_p \geqslant v_p$ ($u_p > v_p$ respectively) for all $p \in M$.

Definition. Define a real operator $\square$ on tensor fields with respect to a given Kaehler metric in the following way:

$$\square = -\tfrac{1}{2}\nabla^*\nabla = \tfrac{1}{2}\left(\nabla_{\bar{E}_i}\nabla_{E_i} + \nabla_{E_i}\nabla_{\bar{E}_i}\right) = \tfrac{1}{2}\left(\nabla_{e_i}\nabla_{e_i}\right),$$

where $\{E_i\}$ is an orthonormal basis of the holomorphic tangent space T_pM, and $\{e_i\}$ is an orthonormal basis of the real tangent space.

Proposition 1. *Let M be an n-dimensional compact Kaehler manifold. Consider the following equation on tensor fields u which have the same type and symmetric properties as curvature tensor*:

$$du/dt = \Box u + F(u),$$

where the Kaehler metric g depends on t, and the smooth function F has the following property (#):

(#) *If $u \geqslant 0$ and there exist two nonzero vectors $X_0, Y_0 \in T_pM$ such that $u_p(X_0, \overline{X_0}, Y_0, \overline{Y_0}) = 0$, then $F(u)_p(X_0, \overline{X_0}, Y_0, \overline{Y_0}) \geqslant 0$.*

If the initial u is nonnegative, then it remains so. Moreover if the initial u is positive somewhere, then the solution u is positive everywhere for $t > 0$.

Proof. It is sufficient to prove the proposition for a short time, thus we consider it in a short interval without specification.

First we define a smooth parallel tensor field u_0 as

$$u_{0_{i\bar{j}k\bar{l}}} = \tfrac{1}{2}(g_{i\bar{j}}g_{k\bar{l}} + g_{i\bar{l}}g_{k\bar{j}}),$$

which is positive everywhere. Then, there exists a positive constant $C \in \mathbf{R}$ such that $Cu_0 \geqslant du_0/dt \geqslant -Cu_0$. Since F is smooth and $u_0 > 0$, there exists a positive constant $D \in \mathbf{R}$ such that

$$F(u) \geqslant F(u + fu_0) - D|f|u_0 \quad \text{for } f \in \mathbf{R},\ |f| \leqslant 1.$$

Let f be a real valued function and $\varepsilon > 0$ a sufficiently small real number. Then,

$$\begin{aligned}\frac{d}{dt}(u + \varepsilon fu_0) &= \Box(u + \varepsilon fu_0) + F(u) + \varepsilon f\frac{d}{dt}u_0 + \varepsilon\left(\frac{d}{dt}f - \Box f\right)u_0 \\ &\geqslant \Box(u + \varepsilon fu_0) + F(u + \varepsilon fu_0) + \varepsilon\left(\frac{d}{dt}f - \Box f - (C + D)|f|\right)u_0.\end{aligned}$$

We choose f to be the solution of the equation

$$\frac{d}{dt}f = \Box f + (C + D)f + 1, \qquad f|_{t=0} = 1.$$

Then $f > 0$ and we get

$$\frac{d}{dt}(u + \varepsilon fu_0) > \Box(u + \varepsilon fu_0) + F(u + \varepsilon fu_0).$$

Here we can prove that $u + \varepsilon fu_0 > 0$. If it is not true, there is the first time $t_0 > 0$ so that it fails to hold because $u|_{t=0} \geqslant 0$ and $f|_{t=0} > 0$. By the definition it follows that, at t_0, $(u + \varepsilon fu_0) \geqslant 0$, and there exist a point $p \in M$ and nonzero vectors $X_0, Y_0 \in T_pM$ such that

$$(u + \varepsilon fu_0)(X_0, \overline{X_0}, Y_0, \overline{Y_0}) = 0.$$

Thus we get, at t_0,

$$F(u + \varepsilon f u_0)\left(X_0, \overline{X_0}, Y_0, \overline{Y_0} \right) \geqslant 0.$$

We extend X_0, Y_0 to be vector fields such that dX_0/dt, ∇X_0, dY_0/dt, $\nabla Y_0 = 0$ at (t_0, p), where ∇ is the covariant derivative with respect to the Kaehler metric at t_0. Then, at (t_0, p),

$$\begin{aligned} 0 &\geqslant \frac{d}{dt}\left[(u + \varepsilon f u_0)\left(X_0, \overline{X_0}, Y_0, \overline{Y_0} \right) \right] \\ &= \left[\frac{d}{dt}(u + \varepsilon f u_0) \right]\left(X_0, \overline{X_0}, Y_0, \overline{Y_0} \right) \\ &> \left[\square(u + \varepsilon f u_0) \right]\left(X_0, \overline{X_0}, Y_0, \overline{Y_0} \right) \\ &= \square\left[(u + \varepsilon f u_0)\left(X_0, \overline{X_0}, Y_0, \overline{Y_0} \right) \right] \geqslant 0. \end{aligned}$$

This is a contradiction. Thus we get that $u + \varepsilon f u_0 > 0$ for all sufficiently small $\varepsilon > 0$, which means $u \geqslant 0$.

The proof of the last statement of the proposition is similar:

$$\frac{d}{dt}(u - f u_0) \geqslant \square(u - f u_0) + F(u - f u_0) + \left(-\frac{d}{dt} f + \square f - (C + D)|f| \right) u_0.$$

We choose the function f so that

$$u - f u_0|_{t=0} \geqslant 0, \quad f|_{t=0} \geqslant 0, \quad f|_{t=0} \not\equiv 0,$$
$$\frac{d}{dt} f = \square f - (C + D) f.$$

Then we get that $u - f u_0 \geqslant 0$ by the above argument, and it is well known that under the above condition, f is positive for $t > 0$ (cf. [6]).

Proof of Theorem 2. Calculating directly we get

$$\begin{aligned} \frac{d}{dt} R_{i\bar{j}k\bar{l}} &= \nabla_k \nabla_{\bar{l}} R_{i\bar{j}} - R_{i\bar{\alpha}} R_{\alpha\bar{j}k\bar{l}}, \\ \nabla_k \nabla_{\bar{l}} R_{i\bar{j}} &= \nabla_k \nabla_{\bar{l}} R_{i\bar{j}\beta\bar{\beta}} = \nabla_k \nabla_{\bar{\beta}} R_{i\bar{j}\beta\bar{l}} \\ &= \nabla_{\bar{\beta}} \nabla_\beta R_{i\bar{j}k\bar{l}} - R_{i\bar{\alpha}k\bar{\beta}} R_{\alpha\bar{j}\beta\bar{l}} + R_{\alpha\bar{j}k\bar{\beta}} R_{i\bar{\alpha}\beta\bar{l}} - R_{k\bar{\alpha}} R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}k\bar{\beta}} R_{i\bar{j}\beta\bar{\alpha}}, \\ \nabla_{\bar{\beta}} \nabla_\beta R_{i\bar{j}k\bar{l}} &= \nabla_\beta \nabla_{\bar{\beta}} R_{i\bar{j}k\bar{l}} + R_{i\bar{\alpha}} R_{\alpha\bar{j}k\bar{l}} - R_{\alpha\bar{j}} R_{i\bar{\alpha}k\bar{l}} + R_{k\bar{\alpha}} R_{i\bar{j}\alpha\bar{l}} - R_{\alpha\bar{l}} R_{i\bar{j}k\bar{\alpha}}, \\ \frac{d}{dt} R_{i\bar{j}k\bar{l}} &= \frac{1}{2}\left(\nabla_{\bar{\beta}} \nabla_\beta R_{i\bar{j}k\bar{l}} + \nabla_\beta \nabla_{\bar{\beta}} R_{i\bar{j}k\bar{l}} \right) \\ &\quad + R_{i\bar{\alpha}\beta\bar{l}} R_{\alpha\bar{j}k\bar{\beta}} - R_{i\bar{\alpha}k\bar{\beta}} R_{\alpha\bar{j}\beta\bar{l}} + R_{i\bar{j}\alpha\bar{\beta}} R_{\beta\bar{\alpha}k\bar{l}} \\ &\quad - \frac{1}{2}\left(R_{i\bar{\alpha}} R_{\alpha\bar{j}k\bar{l}} + R_{\alpha\bar{j}} R_{i\bar{\alpha}k\bar{l}} + R_{k\bar{\alpha}} R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}} R_{i\bar{j}k\bar{\alpha}} \right). \end{aligned}$$

For short, we write the above equation in the following way:

$$\frac{d}{dt}R = \Box R + F(R) - \frac{1}{2}\,\mathrm{Ric} * R.$$

It is obvious that if $R \geqslant 0$ and $R_p(X, \overline{X}, Y, \overline{Y}) = 0$ for some X, $Y \in T_pM$, then $(\mathrm{Ric} * R)(X, \overline{X}, Y, \overline{Y}) = 0$. Thus to prove the theorem it is sufficient to see that F has the property (#).

Let $R \geqslant 0$ and $X, Y \in T_pM$ be unit vectors such that $R(X, X, Y, Y) = 0$. Then considering the second variation we get

$$R(X, \overline{X}, Y', \overline{Y'}) + 2\,\mathrm{Re}\big[R(X, \overline{X'}, Y, \overline{Y'}) + R(X, \overline{X'}, Y', \overline{Y})\big] + R(X', \overline{X'}, Y, \overline{Y}) \geqslant 0,$$

for all $X', Y' \in T_pM$. Thus

$$(\#\#)\qquad \big[|R(X, \overline{X'}, Y, \overline{Y'})| + |R(X, \overline{X'}, Y', \overline{Y})|\big]^2 \leqslant R(X, \overline{X}, Y', \overline{Y'})R(X', \overline{X'}, Y, \overline{Y}),$$

for all $X', Y' \in T_pM$.

$$\begin{aligned} F(R)(X, \overline{X}, Y, \overline{Y}) &= \sum_{i,j}\Big[|R(X, \overline{E_i}, E_j, \overline{Y})|^2 - |R(X, \overline{E_i}, Y, \overline{E_j})|^2 \\ &\qquad + R(X, \overline{X}, E_i, \overline{E_j})R(E_j, \overline{E_i}, Y, \overline{Y})\Big] \\ &\geqslant \sum_{i,j}\Big[-|R(X, \overline{E_i}, Y, \overline{E_j})|^2 \\ &\qquad + R(X, \overline{X}, E_i, \overline{E_j})R(E_j, \overline{E_i}, Y, \overline{Y})\Big], \end{aligned}$$

where $\{E_i\}$ is an orthonormal basis of T_pM.

We divide into two cases.

First case. $X \| Y$. In this case we can assume $X = Y$. Then,

$$F(R)(X, \overline{X}, Y, \overline{Y}) \geqslant \sum_{i,j}\Big[-|R(X, \overline{E_i}, X, \overline{E_j})|^2 + |R(X, \overline{X}, E_i, \overline{E_j})|^2\Big].$$

By the symmetry we can choose $\{E_i\}$ so that

$$R\left(X, \overline{E_i}, X, \overline{E_j}\right) = 0 \quad \text{if } i \neq j.$$

Then, by (# #),

$$F(R)(X, \overline{X}, Y, \overline{Y}) \geqslant \sum_i \left[-\left|R\left(X, \overline{E_i}, X, \overline{E_i}\right)\right|^2 + \left|R\left(X, \overline{X}, E_i, \overline{E_i}\right)\right|^2 \right] \geqslant 0.$$

Second case. $X \nparallel Y$. Because T_pM is three dimensional, there is a unique unit vector E (of cause unique up to constant multiplication) which is orthogonal to X and Y. Then,

$$\begin{aligned} &F(R)(X, \overline{X}, Y, \overline{Y}) \\ &\quad \geqslant -|R(X, \overline{E}, Y, \overline{E})|^2 + \sum_{i,j} R\left(X, \overline{X}, E_i, \overline{E_j}\right) R\left(E_j, \overline{E_i}, Y, \overline{Y}\right), \end{aligned}$$

where we used the fact that if one of the vectors in the expression $R(X, \overline{X}, Y, \overline{Y})$ is replaced by an arbitrary vector, it gives zero. We have two cases.

Case 1. $R(X, \overline{X}, X, \overline{X}) = 0$ or $R(Y, \overline{Y}, Y, \overline{Y}) = 0$.

Case 2. $R(X, \overline{X}, X, \overline{X}) \neq 0$ and $R(Y, \overline{Y}, Y, \overline{Y}) \neq 0$.

Case 1 is easy, for example in the case $R(X, \overline{X}, X, \overline{X}) = 0$, we choose $\{E_i\}$ to be $\{X', Y, E\}$. Then,

$$\sum_{i,j} R\left(X, \overline{X}, E_i, \overline{E_j}\right) R\left(E_j, \overline{E_i}, Y, \overline{Y}\right) = R(X, \overline{X}, E, \overline{E}) R(E, \overline{E}, Y, \overline{Y}),$$

$$\begin{aligned} F(R)(X, \overline{X}, Y, \overline{Y}) &\geqslant -|R(X, \overline{E}, Y, \overline{E})|^2 + R(X, \overline{X}, E, \overline{E}) R(E, \overline{E}, Y, \overline{Y}) \\ &\geqslant 0, \end{aligned}$$

by (# #).

In Case 2 we argue as follows. By (# #) we get that for any complex numbers $s, t \in \mathbf{C}$,

$$\begin{aligned} |R(X, \overline{E}, Y, \overline{E})|^2 &= \left|R\left(X, \overline{E - tY}, Y, \overline{E - sX}\right)\right|^2 \\ &\leqslant R\left(X, \overline{X}, E - sX, \overline{E - sX}\right) R\left(E - tY, \overline{E - tY}, Y, \overline{Y}\right). \end{aligned}$$

Choosing s, t suitably, we get that

$$\begin{aligned} |R(X, \overline{E}, Y, \overline{E})|^2 \leqslant &\left[R(X, \overline{X}, E, \overline{E}) - |R(X, \overline{X}, E, \overline{X})|^2 / R(X, \overline{X}, X, \overline{X}) \right] \\ &\times \left[R(E, \overline{E}, Y, \overline{Y}) - |R(Y, \overline{E}, Y, \overline{Y})|^2 / R(Y, \overline{Y}, Y, \overline{Y}) \right]. \end{aligned}$$

We choose $\{E_i\}$ to be $\{X', Y, E\}$, $X' = aX + bY$.

$$\begin{aligned}
&\sum_{i,j} R\left(X, \overline{X}, E_i, \overline{E_j}\right) R\left(E_j, \overline{E_i}, Y, \overline{Y}\right) \\
&\quad = R(X, \overline{X}, E, \overline{E}) R(E, \overline{E}, Y, \overline{Y}) + R(X, \overline{X}, X', \overline{X'}) R(X', \overline{X'}, Y, \overline{Y}) \\
&\quad + R(X, \overline{X}, E, \overline{X'}) R(X', \overline{E}, Y, \overline{Y}) + R(X, \overline{X}, X', \overline{E}) R(E, \overline{X'}, Y, \overline{Y}) \\
&= R(X, \overline{X}, E, \overline{E}) R(E, \overline{E}, Y, \overline{Y}) + |ab|^2 R(X, \overline{X}, X, \overline{X}) R(Y, \overline{Y}, Y, \overline{Y}) \\
&\quad + 2\,\mathrm{Re}\left[\bar{a} b R(X, \overline{X}, E, \overline{X}) R(Y, \overline{E}, Y, \overline{Y})\right],
\end{aligned}$$

$$\begin{aligned}
F(R)(X, \overline{X}, Y, \overline{Y}) \geqslant & -\left[R(X, \overline{X}, E, \overline{E}) - \frac{|R(X, \overline{X}, E, \overline{X})|^2}{R(X, \overline{X}, X, \overline{X})}\right] \\
& \times\left[R(E, \overline{E}, Y, \overline{Y}) - \left|R(E, \overline{Y}, Y, \overline{Y})\right|^2 / R(Y, \overline{Y}, Y, \overline{Y})\right] \\
& + R(X, \overline{X}, E, \overline{E}) R(E, \overline{E}, Y, \overline{Y}) \\
& - \frac{|R(X, \overline{X}, E, \overline{X})|^2}{R(X, \overline{X}, X, \overline{X})} \frac{|R(E, \overline{Y}, Y, \overline{Y})|^2}{R(Y, \overline{Y}, Y, \overline{Y})} \\
= & \; R(X, \overline{X}, E, \overline{E}) \left|R(E, \overline{Y}, Y, \overline{Y})\right|^2 / R(Y, \overline{Y}, Y, \overline{Y}) \\
& + \left[\left|R(X, \overline{X}, E, \overline{X})\right|^2 / R(X, \overline{X}, X, \overline{X})\right] R(E, \overline{E}, Y, \overline{Y}) \\
& - 2 \frac{|R(X, \overline{X}, E, \overline{X})|^2}{R(X, \overline{X}, X, \overline{X})} \frac{|R(E, \overline{Y}, Y, \overline{Y})|^2}{R(Y, \overline{Y}, Y, \overline{Y})} \\
\geqslant & \; 0.
\end{aligned}$$

We state several properties on the solutions of Hamilton's equation which can be proved similarly. We always assume that the manifold M is a three-dimensional compact Kaehler manifold and the initial metric has nonnegative bisectional curvature, thus the solution metric remains so.

Proposition 2. *For the solution metric at* $t > 0$, *if* $R(X, \overline{X}, Y, \overline{Y}) = 0$ *with some* $0 \neq X, Y \in T_pM$, *then* $R(X, \bar{\cdot}, \cdot, \overline{Y}) = 0$.

Proof. Extend X, Y to be vector fields as in the proof of Proposition 1. Then, from the proof,

$$\begin{aligned}
0 = \frac{d}{dt}[R(X, \overline{X}, Y, \overline{Y})] &= \Box[R(X, \overline{X}, Y, \overline{Y})] + F(R)(X, \overline{X}, Y, \overline{Y}) \\
&\geqslant \sum_{i,j} \left|R\left(X, \overline{E_i}, E_j, \overline{Y}\right)\right|^2.
\end{aligned}$$

Proposition 3. *If the initial metric has quasipositive Ricci curvature then for* $t > 0$ *the solution metric has positive Ricci curvature.*

Proposition 4. *If the initial metric has the property* $(**)$ *at some point then for* $t > 0$ *the solution metric has the property* $(**)$ *everywhere. In particular* M *must be biholomorphic to either* P^3 *or* Q^3.

Corollary. *If* M *is biholomorphic to neither* P^3 *nor* Q^3, *then for every point* $p \in M$ *there exists a nonzero vector* $X \in T_pM$ *such that*

$$(\&) \qquad R(X, \overline{X}, Y, \overline{Y}) = 0 \quad \text{for all } Y \perp X,\ Y \in T_pM.$$

Proposition 5. *If the initial metric has a point* $p \in M$ *such that*

$$R(X, \overline{X}, Y, \overline{Y}) + R(Y, \overline{Y}, Z, \overline{Z}) + R(Z, \overline{Z}, X, \overline{X}) > 0$$

for any orthonormal basis $\{X, Y, Z\}$ *of* T_pM, *then the solution metric for* $t > 0$ *has the above property everywhere.*

5. The classification

This section is devoted to the proof of the following classification theorem.

Theorem 3. *Let* M *be a three-dimensional compact Kaehler manifold of nonnegative bisectional curvature. If* M *has quasipositive Ricci curvature, then* M *is biholomorphic to one of the following*: P^3, Q^3, $P^1 \times P^2$, $P^1 \times P^1 \times P^1$.

Remark. It is known by [3] that in the above case the assumption of quasipositive Ricci curvature is equivalent to positivity of the first Chern class.

Proof of Theorem 3. We assume, besides the assumption of the theorem, that M is biholomorphic to neither P^3 nor Q^3. We understand that when we mention metric, curvature and so on, we always mean those of a solution metric of Hamilton's equation. Especially we have positive Ricci curvature and property (&) is satisfied by some unit vector $X \in T_pM$ for every $p \in M$.

We have two cases at each point $p \in M$.

Case 1. The unit vector $X \in T_pM$ satisfying (&) is unique up to constant multiplication.

Case 2. The unit vector $X \in T_pM$ satisfying (&) is not unique up to constant multiplication.

First we look at Case 2. By Proposition 2, for such a unit vector $X \in T_pM$ we have that

$$R(X, \bar{\cdot}, \cdot, \overline{Y}) = 0 \quad \text{for all } Y \perp X,\ Y \in T_pM.$$

Let X' be another unit vector satisfying (&) which is linearly independent of X. Then since $\dim_{\mathbf{C}} T_pM = 3$, there is a unit vector Y such that

$$R(X, \bar{\cdot}, \cdot, \overline{Y}) = R(X', \bar{\cdot}, \cdot, \overline{Y}) = 0, \qquad Y \perp X, X'.$$

This means that Y also satisfies (&). Replacing X' by Y in the above argument, we obtain an orthonormal basis $\{X, Y, Z\}$ which consists of the vectors satisfying (&). Moreover the orthonormal basis of such a property is unique up to constant multiplication and every vector satisfying (&) is a constant multiplication of some element of the orthonormal basis. To see this it is sufficient to prove the following fact.

Fact. Let $\{E_i\}$ be a fixed orthonormal basis which consists of the vectors satisfying (&), and let X be an arbitrary vector satisfying (&). Then X is either parallel or orthogonal to each E_i.

In the above argument we have already seen that for each E_i there exists an orthonormal basis $\{E_i, Y, Z\}$ which consists of the vectors satisfying (&), and $X = aE_i + bZ$ with some complex numbers a, b. Because $\bar{b}E_i - \bar{a}Z$ is orthogonal to X, we get that

$$R\left(aE_i + bZ, \bar{\cdot}, \cdot, \overline{\bar{b}E_i - \bar{a}Z}\right) = 0, \quad abR\left(E_i, \bar{\cdot}, \cdot, \overline{E_i}\right) = abR(Z, \bar{\cdot}, \cdot, \bar{Z}).$$

Thus

$$\begin{aligned} ab\,\mathrm{Ric}\left(E_i, \overline{E_i}\right) &= abR\left(E_i, \overline{E_i}, E_i, \overline{E_i}\right) + abR\left(E_i, \overline{E_i}, Y, \bar{Y}\right) \\ &\quad + abR\left(E_i, \overline{E_i}, Z, \bar{Z}\right) \\ &= abR\left(Z, \overline{E_i}, E_i, \bar{Z}\right) = 0. \end{aligned}$$

Since Ricci curvature is positive, $ab = 0$, which is what we wanted to prove.

Thus by Proposition 5 we know that either Case 1 holds everywhere or Case 2 holds everywhere, and in both cases vectors with the property (&) are unique in the appropriate sense. Here we can prove that in both cases such vectors make differentiable distributions. To see this by the implicit function theorem it is sufficient to prove that if X is a unit vector with the property (&), the derivative of $R(X, \bar{\cdot}, \cdot, \overline{Y - \langle Y, \bar{X}\rangle X})$ in the direction $X' \perp X$ is zero for all Y, then X' is zero, where $\langle\ ,\ \rangle$ is the inner product. Setting the derivative to be zero we get that

$$\begin{aligned} R\left(X', \bar{\cdot}, \cdot, \overline{Y - \langle Y, \bar{X}\rangle X}\right) &+ R\left(X, \bar{\cdot}, \cdot, \overline{-\langle Y, \overline{X'}\rangle X}\right) \\ &+ R\left(X, \bar{\cdot}, \cdot, \overline{-\langle Y, \bar{X}\rangle X'}\right) = 0. \end{aligned}$$

Taking $Y = X'$, and substituting X, we have that

$$\langle X', \overline{X'}\rangle R(X, \bar{X}, X, \bar{X}) = 0.$$

Using the positivity of Ricci curvature again we get that X' is zero.

Next we show that these distributions are parallel. If once we know this, we can prove the theorem easily. Since M has positive Ricci curvature, M is simply connected (cf. [9]), we can apply the de Rham decomposition theorem and we get that in Case 1, M is holomorphically isometric to $P^1 \times N^2$ where P^1, N are equipped with the metrics of nonnegative bisectional curvature with positive Ricci curvature, and in Case 2, M is biholomorphic to $P^1 \times P^1 \times P^1$. Then we apply Theorem A and get that N is biholomorphic to either P^2 or $P^1 \times P^1$. This is the desired result.

The proof of parallelism of the distributions is actually the same in both cases; we consider only Case 1. Since parallelism is a local property, we work locally from now on. Let $X \in T_pM$ be a unit vector with the property (&) and $Y \in T_pM$ be an arbitrary vector orthogonal to X. We extend them to be vector fields. Then we have at p that

$$
\begin{aligned}
0 &= \frac{d}{dt}[R(X, \bar{X}, Y, \bar{Y})] = \left[\frac{d}{dt}R\right](X, \bar{X}, Y, \bar{Y}) \\
&= [\Box R](X, \bar{X}, Y, \bar{Y}) + F(R)(X, \bar{X}, Y, \bar{Y}) - \frac{1}{2}[\mathrm{Ric} * R](X, \bar{X}, Y, \bar{Y}) \\
&\geqslant [\Box R](X, \bar{X}, Y, \bar{Y}),
\end{aligned}
$$

$$
\begin{aligned}
0 &\leqslant \Box[R(X, \bar{X}, Y, \bar{Y})] \\
&= [\Box R](X, \bar{X}, Y, \bar{Y}) + [\nabla R](\nabla X, \bar{X}, Y, \bar{Y}) + [\nabla R](X, \overline{\nabla X}, Y, \bar{Y}) \\
&\quad + [\nabla R](X, \bar{X}, \nabla Y, \bar{Y}) + [\nabla R](X, \bar{X}, Y, \overline{\nabla Y}) + R(\Box X, \bar{X}, Y, \bar{Y}) \\
&\quad + R(X, \overline{\Box X}, Y, \bar{Y}) + R(X, \bar{X}, \Box Y, \bar{Y}) + R(X, \bar{X}, Y, \overline{\Box Y}) \\
&\quad + R(\nabla X, \overline{\nabla X}, Y, \bar{Y}) + R(\nabla X, \bar{X}, \nabla Y, \bar{Y}) + R(\nabla X, \bar{X}, Y, \overline{\nabla Y}) \\
&\quad + R(X, \overline{\nabla X}, \nabla Y, \bar{Y}) + R(X, \overline{\nabla X}, Y, \overline{\nabla Y}) + R(X, \bar{X}, \nabla Y, \overline{\nabla Y}) \\
&\leqslant [\nabla R](\nabla X, \bar{X}, Y, \bar{Y}) + [\nabla R](X, \overline{\nabla X}, Y, \bar{Y}) + [\nabla R](X, \bar{X}, \nabla Y, \bar{Y}) \\
&\quad + [\nabla R](X, \bar{X}, Y, \overline{\nabla Y}) + R(\nabla X, \overline{\nabla X}, Y, \bar{Y}) + R(X, \bar{X}, \nabla Y, \overline{\nabla Y}),
\end{aligned}
$$

where the repeated ∇ means the summation over real orthonormal basis, and we used the fact that $R(X, \bar{\cdot}, Y, \bar{\cdot}) = 0$ which is implied by (# #) and (&). The above inequality holds for any extensions X, Y; thus if one of the vectors in the expression $[\nabla R](X, \bar{X}, Y, \bar{Y})$ is replaced by an arbitrary vector, we get zero.

Next we choose X as a vector field satisfying (&) at each point and Y as any vector field which is orthogonal to X at each point. Then we get that for any vector field Z,

$$
\begin{aligned}
0 &= \nabla[R(Z, \bar{X}, Y, \bar{Y})] \\
&= [\nabla R](Z, \bar{X}, Y, \bar{Y}) + R(\nabla Z, \bar{X}, Y, \bar{Y}) + R(Z, \overline{\nabla X}, Y, \bar{Y}) \\
&\quad + R(Z, \bar{X}, \nabla Y, \bar{Y}) + R(Z, \bar{X}, Y, \overline{\nabla Y}).
\end{aligned}
$$

We know that, except $R(Z, \overline{\nabla X}, Y, \bar{Y})$, all other terms vanish; thus $R(Z, \overline{\nabla X}, Y, \bar{Y}) = 0$ for any vector Z, and any vector $Y \perp X$.

Let W be the orthogonal projection of ∇X to the orthogonal complement of the space generated by X, and $\{X, Y, Z\}$ be the orthonormal basis. Then

$$\begin{aligned}\mathrm{Ric}(W, \bar{W}) &= R(W, \bar{W}, X, \bar{X}) + R(W, \bar{W}, Y, \bar{Y}) + R(W, \bar{W}, Z, \bar{Z}) \\ &= R(W, \overline{\nabla X}, Y, \bar{Y}) + R(W, \overline{\nabla X}, Z, \bar{Z}) \\ &= 0.\end{aligned}$$

We get that $W = 0$. This means that X gives rise to a parallel distribution.

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